All admissible k-tuples arise and persist in Eratosthenes sieve

Joint Mathematics Meetings 2025
Seattle, WA

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https://www.primegaps.info

Patterns among the Primes

Eratosthenes sieve as a discrete dynamic system

Conjectures about constellations between primes

***k*-tuple conjecture (1923)**
Every admissible constellation occurs infinitely often as a constellation of gaps between primes.

Hardy & Littlewood estimate (1923)
Every admissible constellation occurs infinitely often as a constellation of gaps between primes, with relative frequency of occurrence of

$$\prod_{p>2} \left(\frac{p}{p-1}\right)^J \frac{p-\nu}{p-1}$$

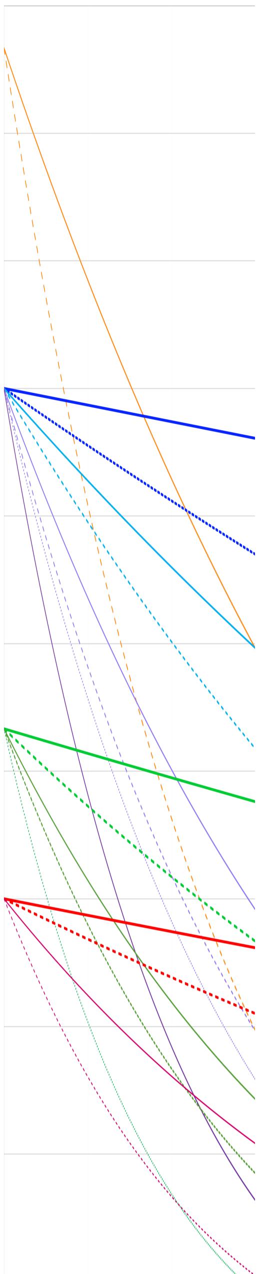
Polignac's conjecture (1849)
For every even number *2n*, the gap *g=2n* occurs infinitely often as a gap between primes.

Twin prime conjecture
The gap *g=2* occurs infinitely often as a gap between primes.

Hardy & Littlewood estimate (1923)
*citing Sylvester (1871)

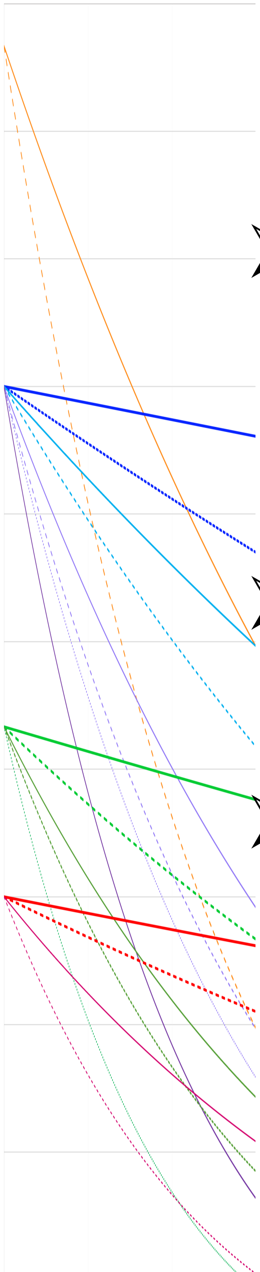
For every even number *2n*, the gap *g=2n* occurs infinitely often as a gap between primes, with relative frequency of occurrence of

$$\prod_{\text{odd } p \mid g} \frac{p-1}{p-2}$$



Theorem:

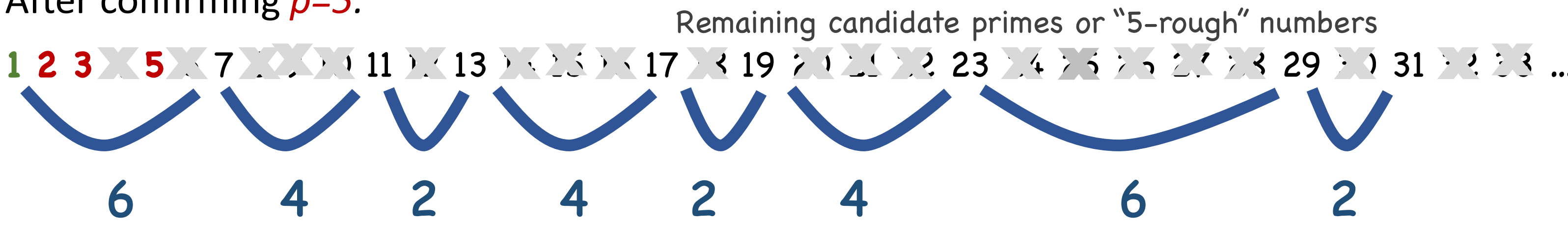
All admissible instances of every admissible constellation occur in Eratosthenes sieve.



*Context from **Patterns among the Primes***

- Eratosthenes sieve

After confirming *p=5:*



- Cycles of gaps

$$\mathcal{G}(5^{\#}) = 6\ 4\ 2\ 4\ 2\ 4\ 6\ 2$$

$\mathcal{G}(p^{\#})$ has length $\phi(p^{\#})$ and span $p^{\#}$.

Polignac (1849)

- Recursion

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Context from ***Patterns among the Primes***

- ## Eratosthenes sieve

After confirming *p=5:*

Remaining candidate primes or "5-rough" numbers

1 2 3 5 7 11 13 17 19 23 29 31 …

Unit — Confirmed primes

- ## Cycles of gaps

$$\mathcal{G}(5^\#) = 6\ 4\ 2\ 4\ 2\ 4\ 6\ 2$$

$\mathcal{G}(p^\#)$ has length $\phi(p^\#)$ and span $p^\#$.

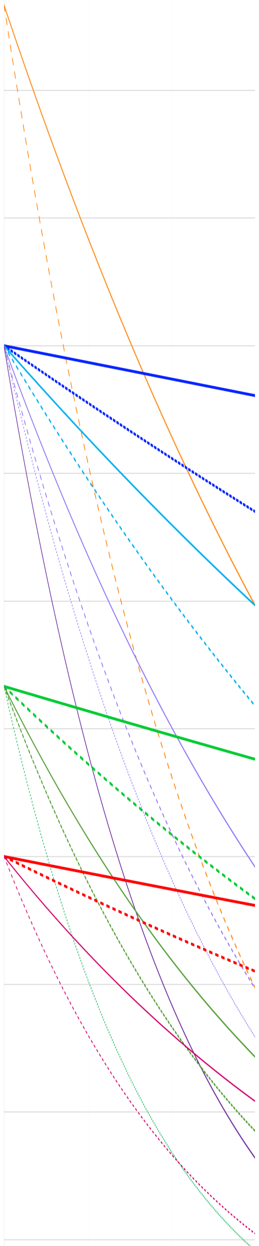
- ## Recursion $\mathcal{G}(p_k{}^\#) \longrightarrow \mathcal{G}(p_{k+1}{}^\#)$

For p_k we have the cycle of gaps $\mathcal{G}(p_k{}^\#) = g_1\, g_2\, g_3 \dots g_N$

R1. The next prime $p_{k+1} = g_1+1$

R2. Concatenate p_{k+1} copies of $\mathcal{G}(p_k{}^\#)$

R3. Fusions: Add together the gaps g_1+g_2 and thereafter add adjacent gaps at the running sums indicated by the elementwise product $p_{k+1} * \mathcal{G}(p_k{}^\#)$



Constellations of gaps

A *constellation* of gaps of length J is a sequence of J consecutive gaps.

There is a natural correspondence between constellations of length J and k-tuples with $k = J+1$.

$$s = \quad g_1 \quad g_2 \quad g_3 \quad g_4 \quad \ldots \quad g_J$$

$$\gamma_0 \quad \gamma_1 \quad \gamma_2 \quad \gamma_3 \quad \gamma_4 \quad \ldots \quad \gamma_J$$

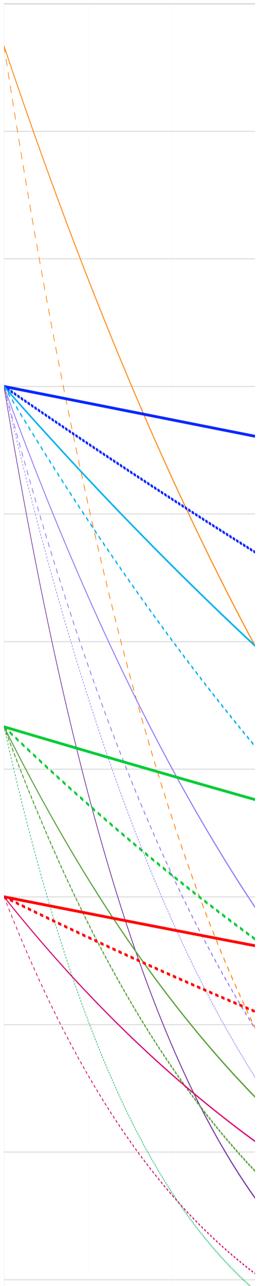
An ***instance*** of a constellation s is fixed by identifying γ_0.

The remaining generators are

$$\gamma_j = \gamma_{j-1} + g_j$$

An instance γ_0 of s is ***admissible*** for the prime p iff

$$\gamma_j \bmod p \neq 0 \text{ for all } j=0,\ldots,J$$



Admissible constellations

For a prime *p* and a constellation *s,* we define

ν_p to be the number of distinct residues mod *p* covered by any *k*-tuple corresponding to *s.*

A constellation *s* is ***admissible for a prime p*** iff $\nu_p < p$.

p	***s = 2,10,2,10,2***	γ_i **mod *p***	***admissible*** γ_0 **mod *p***
3	$\nu_3 = 2$	{0, 2}	2
5	$\nu_5 = 4$	{0,1,2,4}	2
7	$\nu_7 = 4$	{0,2,3,5}	1,3,6

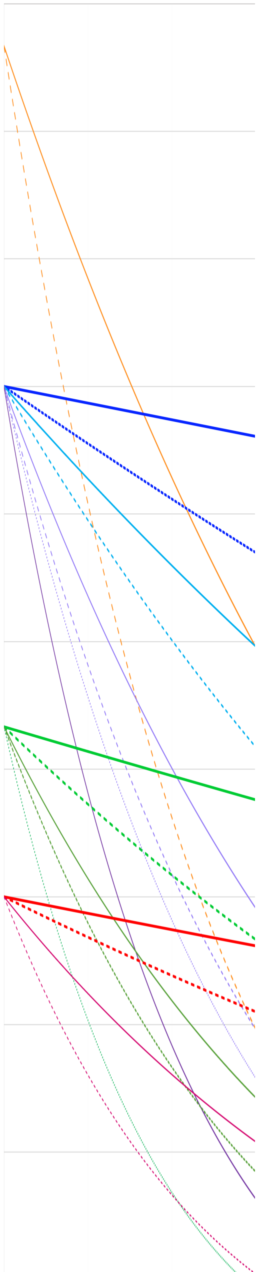
> *Lemma:* There are $p - \nu_p$ values of γ_0 mod *p* for admissible instances of *s*.

A constellation *s* is ***admissible*** iff *s* is admissible for all primes *p.*

$w_{s,J}(\infty)$: asymptotic relative populations

The population of s in $\mathcal{G}(p^\#)$ is

$$n_{s,J}(p^\#) = w_{s,J}(p^\#) \cdot \prod_{J+1<q\leq p} (q-J-1)$$

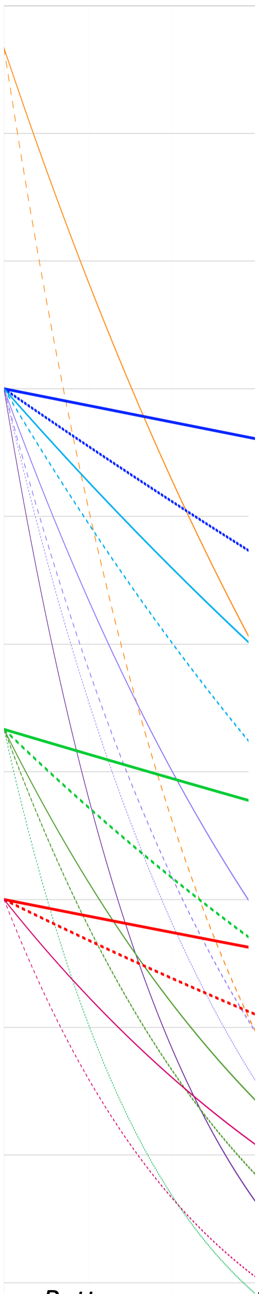


<u>*Theorem*</u>*:*

Let *s* be an admissible constellation of length *J*, and let *Q* be the product of the odd primes that divide a span between boundary fusions in *s.*

Then the asymptotic relative population of *s* in $\mathcal{G}(p^\#)$ is

$$w_{s,J}(\infty) = \prod_{q\leq J+1} (q-\nu_q) \cdot \prod_{q>J+1,\ q|Q} \frac{q-\nu_q}{q-J-1}$$



Consecutive Primes in Arithmetic Progression

Corollary:

Let *s* be an admissible repetition of the gap *g* of length *J*, and let $Q = \prod_{\text{odd } q|g} q$.

Then the asymptotic relative population of *s* in $\mathcal{G}(p^\#)$ is

$$w_{s,J}(\infty) = \frac{\phi(Q)}{\prod_{q|Q}(q - J - 1)}$$

An illustrative example to outline the proof

ν_p is the number of distinct residues mod p

covered by any k-tuple corresponding to s.

p	$s = 2,10,2,10,2$	γ_i mod p	admissible γ_0
3	$\nu_3 = 2$	{0, 2}	2
5	$\nu_5 = 4$	{0,1,2,4}	2
7	$\nu_7 = 4$	{0,2,3,5}	**1,3,6**

$\mathcal{G}(5^\#)$ 6 4 2 4 **2 4 6 2** **6 4 2** 4 2 4 6 2

$\gamma_0=17$

Driving term for s

$\mathcal{G}(7^\#)$

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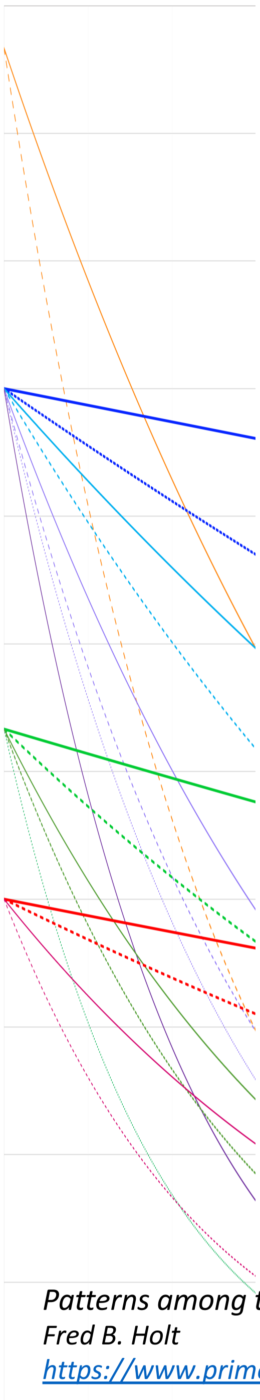

Illustrative example

ν_p is the number of distinct residues mod *p*

covered by any *k*-tuple corresponding to *s.*

p	***s = 2,10,2,10,2***	**γ_i mod *p***	***admissible γ_0***
3	$\nu_3 = 2$	*{0, 2}*	*2*
5	$\nu_5 = 4$	*{0,1,2,4}*	*2*
7	$\nu_7 = 4$	*{0,2,3,5}*	***1,3,6***

$\mathcal{G}(5^\#)$ 6 4 2 4 **2 4 6 2 6 4 2** 4 2 4 6 2

$\gamma_0=17$

Driving term for s

$\mathcal{G}(7^\#)$

$5^\# = 2 \bmod 7$

R2 *replication*

$\gamma_0+0 \cdot 5^\# = 3 \bmod 7$ **2 4 6 2 6 4 2**

$\gamma_0+1 \cdot 5^\# = 5 \bmod 7$ ~~2 4 6 2 6 4 2~~

$\gamma_0+2 \cdot 5^\# = 0 \bmod 7$ ~~2 4 6 2 6 4 2~~

$\gamma_0+3 \cdot 5^\# = 2 \bmod 7$ ~~2 4 6 2 6 4 2~~

$\gamma_0+4 \cdot 5^\# = 4 \bmod 7$ ~~2 4 6 2 6 4 2~~

$\gamma_0+5 \cdot 5^\# = 6 \bmod 7$ **2 4 6 2 6 4 2**

$\gamma_0+6 \cdot 5^\# = 1 \bmod 7$ **2 4 6 2 6 4 2** → **2 10 2 10 2**

R3 *fusions*

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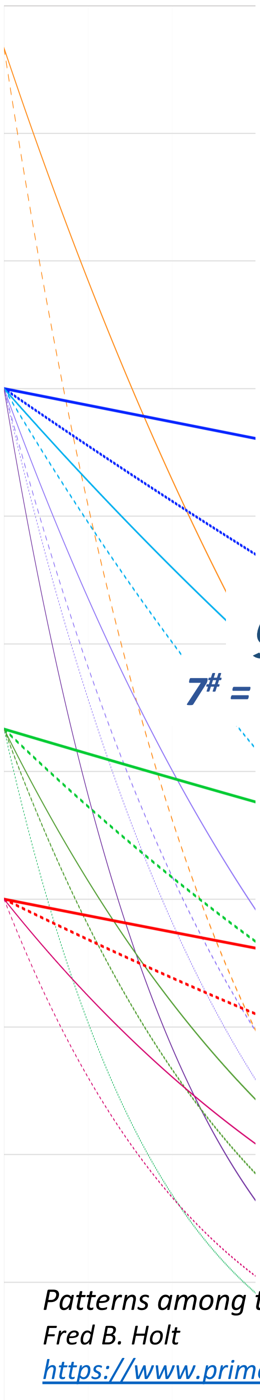

Admissible instances

p	$s = 2,10,2,10,2$	γ_i mod p	admissible γ_0
7	$\nu_7 = 4$	$\{0,2,3,5\}$	1,3,6
11	$\nu_{11} = 5$	$\{0,1,2,3,4\}$	**1,2,3,4,5,6**

$\mathcal{G}(7^\#)$	$\gamma_0 + 0 \cdot 5^\# = 3 \bmod 7$ $= 6 \bmod 11$	$\gamma_0 + 5 \cdot 5^\# = 6 \bmod 7$ $= 2 \bmod 11$	$\gamma_0 + 6 \cdot 5^\# = 1 \bmod 7$ $= 10 \bmod 11$
	2 4 6 2 6 4 2	**2 4 6 2 6 4 2**	**2 10 2 10 2**

$+ 0 \cdot 7^\#$ 6 Driving terms **2** 10

$\mathcal{G}(11^\#)$

$7^\# = 1 \bmod 11$

Driving terms 2,10,2,10,2 ← 2 4 6 2,10,2 / 2,10,2 6 4 2 ← 2 4 6 2 6 4 2

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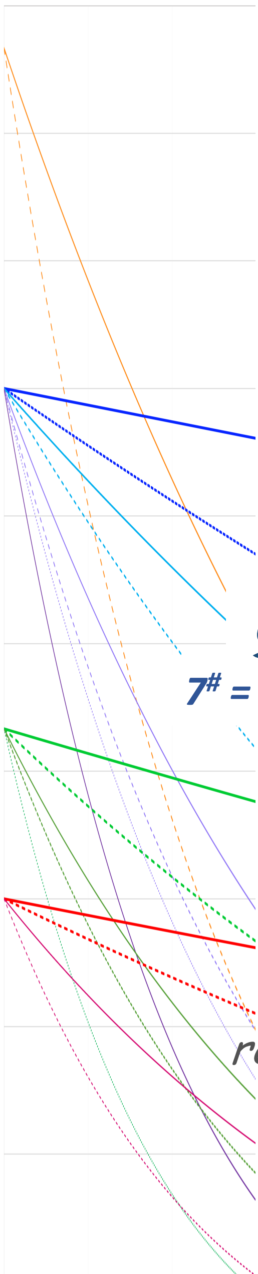

Admissible instances

p	$s = 2,10,2,10,2$	$\gamma_i \bmod p$	admissible γ_0
7	$\nu_7 = 4$	$\{0,2,3,5\}$	1,3,6
11	$\nu_{11} = 5$	$\{0,1,2,3,4\}$	**1,2,3,4,5,6**

$\mathcal{G}(7^\#)$

$\gamma_0+0\cdot 5^\# = 3 \bmod 7$ $= 6 \bmod 11$	$\gamma_0+5\cdot 5^\# = 6 \bmod 7$ $= 2 \bmod 11$	$\gamma_0+6\cdot 5^\# = 1 \bmod 7$ $= 10 \bmod 11$
2 4 6 2 6 4 2	**2 4 6 2 6 4 2**	**2 10 2 10 2**

$\mathcal{G}(11^\#)$

$7^\# = 1 \bmod 11$

R2 replication

R3 fusions

		Driving terms	
$+ 0\cdot 7^\#$	6	2	10
$+ 1\cdot 7^\#$	7	3	0
$+ 2\cdot 7^\#$	8	4	1
$+ 3\cdot 7^\#$	9	5	2
$+ 4\cdot 7^\#$	10	6	3
$+ 5\cdot 7^\#$	0	7	4
$+ 6\cdot 7^\#$	1	8	5
$+ 7\cdot 7^\#$	2	9	6
$+ 8\cdot 7^\#$	3	10	7
$+ 9\cdot 7^\#$	4	0	8
$+ 10\cdot 7^\#$	5	1	9

Driving terms ⬠ 2,10,2,10,2 ← ⬡ 2 4 6 2,10,2 / 2,10,2 6 4 2 ← ⬡ 2 4 6 2 6 4 2

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Admissible instances

$\mathcal{G}(7^\#)$: light green circle $\gamma_0+0\cdot5^\#$; pale circle $\gamma_0+5\cdot5^\#$; dark green circle $\gamma_0+6\cdot5^\#$

p	$s = 2,10,2,10,2$	$\gamma_i \bmod p$	admissible γ_0
11	$\nu_{11} = 5$	$\{0,1,2,3,4\}$	1,2,3,4,5,6
13	$\nu_{13} = 5$	$\{0,1,2,11,12\}$	**3,4,5,6,7,8,9,10**

$\mathcal{G}(11^\#)$

$\mathcal{G}(13^\#)$

$11^\# = 9 \bmod 13$

	$+0\cdot7^\#$		$+1\cdot7^\#$	$+2\cdot7^\#$		$+3\cdot7^\#$		$+4\cdot7^\#$		$+5\cdot7^\#$	$+6\cdot7^\#$		$+7\cdot7^\#$		$+8\cdot7^\#$	$+9\cdot7^\#$	$+10\cdot7^\#$	
	$\gamma_0+0\cdot5^\#$	$\gamma_0+5\cdot5^\#$	$\gamma_0+5\cdot5^\#$	$\gamma_0+5\cdot5^\#$	$\gamma_0+6\cdot5^\#$	$\gamma_0+5\cdot5^\#$	$\gamma_0+6\cdot5^\#$	$\gamma_0+5\cdot5^\#$	$\gamma_0+6\cdot5^\#$	$\gamma_0+6\cdot5^\#$	$\gamma_0+0\cdot5^\#$	$\gamma_0+6\cdot5^\#$	$\gamma_0+0\cdot5^\#$	$\gamma_0+6\cdot5^\#$	$\gamma_0+0\cdot5^\#$	$\gamma_0+0\cdot5^\#$	$\gamma_0+0\cdot5^\#$	$\gamma_0+5\cdot5^\#$
$+0\cdot11^\#$	4	11	0	2	**6**	4	**8**	**6**	**10**	12	3	1	5	**3**	**7**	9	11	5
$+1\cdot11^\#$	0	**7**	9	11	2	0	**4**	2	**6**	**8**	12	**10**	1	12	3	5	7	1
$+2\cdot11^\#$	9	3	5	**7**	11	9	0	11	2	**4**	8	**6**	10	**8**	12	1	3	10
$+3\cdot11^\#$	5	12	1	3	**7**	5	**9**	**7**	11	0	4	2	6	**4**	8	10	12	**6**
$+4\cdot11^\#$	1	8	10	12	**3**	1	**5**	3	**7**	**9**	0	11	2	0	4	**6**	8	2
$+5\cdot11^\#$	10	4	**6**	8	12	10	1	12	**3**	**5**	9	**7**	11	**9**	0	2	4	11
$+6\cdot11^\#$	**6**	0	2	4	**8**	**6**	**10**	8	12	1	5	**3**	**7**	**5**	9	11	0	**7**
$+7\cdot11^\#$	2	9	11	0	**4**	2	**6**	4	**8**	**10**	1	12	3	1	5	**7**	9	3
$+8\cdot11^\#$	11	5	**7**	9	0	11	2	0	**4**	**6**	10	**8**	12	**10**	1	3	5	12
$+9\cdot11^\#$	**7**	1	3	5	**9**	7	11	9	0	2	**6**	**4**	8	**6**	10	12	1	8
$+10\cdot11^\#$	3	10	12	1	**5**	3	**7**	5	**9**	11	2	0	4	2	**6**	8	10	4
$+11\cdot11^\#$	12	6	8	10	1	12	**3**	1	**5**	**7**	11	**9**	0	11	2	4	**6**	0
$+12\cdot11^\#$	8	2	4	**6**	**10**	8	12	10	1	**3**	**7**	**5**	9	**7**	11	0	2	9

Driving terms: (pentagon) 2,10,2,10,2 ← (hexagon) 2 4 6 2,10,2 / 2,10,2 6 4 2 ← (heptagon) 2 4 6 2 6 4 2

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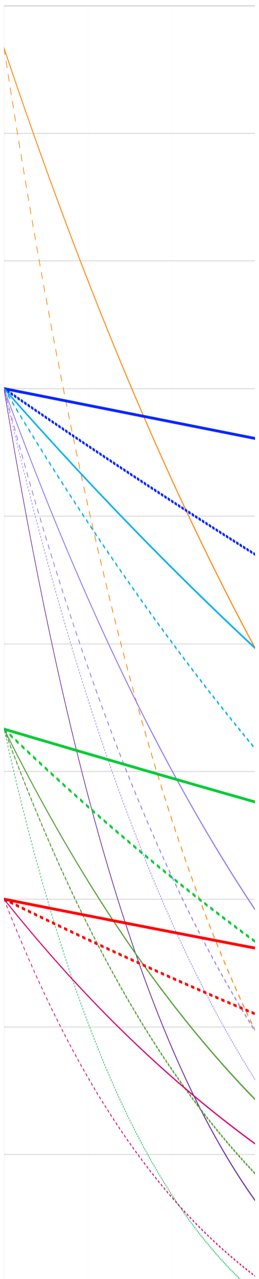
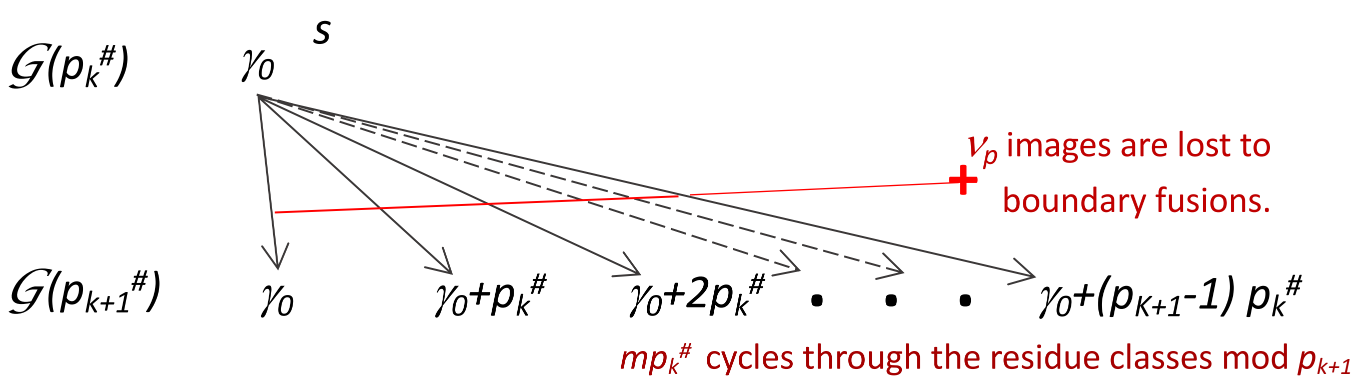


Generalizing, we can prove --



Lemma: For an admissible constellation *s*,

let γ_0 be an admissible instance of *s* in $\mathcal{G}(p_0^\#)$.

For **every** admissible instance of *s* in $\mathcal{G}(p_k^\#)$,

$$\gamma_0 + m_1 p_0^\# + m_2 p_1^\# \ldots + m_k p_{k-1}^\#$$

there are $(p_{k+1} - \nu_p)$ choices of m_{k+1} such that

$$\gamma_0 + m_1 p_0^\# + m_2 p_1^\# \ldots + m_k p_{k-1}^\# + m_{k+1} p_k^\#$$

is an admissible instance of *s* in $\mathcal{G}(p_{k+1}^\#)$,

and these have distinct residues mod $p_{k+1}^\#$.

The relative populations $w_{s,J}(p^\#)$ in $\mathcal{G}(p^\#)$ support the *k*-tuple conjecture

_k_-tuple conjecture (1923)
Every admissible constellation occurs infinitely often as a constellation of gaps between primes.

Hardy & Littlewood estimate (1923)
Every admissible constellation occurs infinitely often as a constellation of gaps between primes, with relative frequency of occurrence of

$$\prod_{p>2} \left(\frac{p}{p-1}\right)^J \frac{p-\nu}{p-1}$$

<u>*Theorem:*</u> Every admissible constellation *s* of length *J* arises and persists in $\mathcal{G}(p^\#)$, with asymptotic relative population

$$w_{s,J}(\infty) = \prod_{q \le J+1} (q - \nu_q) \cdot \prod_{q|Q,\, q>J+1} \frac{q-\nu_q}{q-J-1}$$

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